NCEA Level 2 · AS 91261: Algebra Toolkit

Name: _____ Date: _____

Kia ora! This standard is all about the algebra machinery: simplifying with indices, factorising, and solving quadratic, exponential and log equations. Work through each section in order, check every answer against the worked solution and note the grade level you're aiming for.

Revision summary

- Index laws: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, $a^0 = 1$, $a^{-n} = 1/a^n$, $a^{(1/n)} = \sqrt[n]{a}$.
- Logarithm definition: if $a^x = b$ then $\log_a b = x$. Log laws: $\log(xy) = \log x + \log y$, $\log(x/y) = \log x - \log y$, $\log(x^n) = n \log x$.
- Quadratic formula: for $ax^2 + bx + c = 0$, $x = (-b \pm \sqrt{(b^2 - 4ac)}) \div 2a$.
- Discriminant $\Delta = b^2 - 4ac$: $\Delta > 0$ two real roots, $\Delta = 0$ one repeated root, $\Delta < 0$ no real roots.
- Factorising: common factor first, then quadratics $x^2 + bx + c = (x + p)(x + q)$ where $p + q = b$ and $pq = c$; difference of two squares $a^2 - b^2 = (a - b)(a + b)$.
- Exponential equations: write both sides with the same base, then equate the powers.
- Log equations: combine to a single log, rewrite in index form, then solve, and always check answers against the domain (you can only take logs of positive numbers).

Key skills

1. Simplify $(3a^2b)^3 \times 2ab^2$.

2. Factorise $x^2 + 3x - 10$.

3. Solve $2^x = 32$.

4. Evaluate $\log_3 81$.

Achieved

5. Solve $x^2 - 5x + 6 = 0$ by factorising.

6. Simplify $(12x^5y^3) \div (4x^2y^5)$.

7. Use the quadratic formula to solve $2x^2 + 3x - 4 = 0$. Give exact answers.

8. Solve 3^(x+1) = 81.

Merit

9. A rectangular deck at a bach in Raglan has a length 3 m longer than its width. The area of the deck is 40 m². Form an equation and find the dimensions of the deck.

10. Write $3/(x-2) + 2/(x+1)$ as a single simplified fraction.

11. Solve $9^x = 27^(x-2)$.

Excellence

12. For what values of k does the equation $x^2 + kx + 4 = 0$ have two distinct real roots? Justify your answer.

13. Solve $\log_2 x + \log_2(x-2) = 3$, showing why any invalid solution must be rejected.

14. Prove that the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

Answer key (for parents and tutors)

1. Simplify $(3a^2b)^3 \times 2ab^2$.
   $(3a^2b)^3 = 27a^6b^3$. Multiply by $2ab^2$: $27a^6b^3 \times 2ab^2 = 54a^7b^5$.
   Achieved: applies index laws to a product raised to a power.

2. Factorise $x^2 + 3x - 10$.
   Need two numbers with product -10 and sum $+3$: they are $+5$ and -2. So $x^2 + 3x - 10 = (x + 5)(x - 2)$.
   Achieved: factorises a monic quadratic.

3. Solve $2^x = 32$.
   $32 = 2^5$, so $2^x = 2^5$, therefore $x = 5$.
   Achieved: solves an exponential by matching bases.

4. Evaluate $\log_3 81$.
   $81 = 3^4$, so $\log_3 81 = 4$.
   Achieved: applies the definition of a logarithm.

5. Solve $x^2 - 5x + 6 = 0$ by factorising.
   Two numbers with product 6 and sum -5 are -2 and -3. So $(x - 2)(x - 3) = 0$, giving $x = 2$ or $x = 3$.
   Achieved: selects factorising to solve a quadratic.

6. Simplify $(12x^5y^3) \div (4x^2y^5)$.
   $12 \div 4 = 3$. $x^5 \div x^2 = x^3$. $y^3 \div y^5 = y^{-2} = 1/y^2$. So the expression $= 3x^3/y^2$.
   Achieved: applies index laws including a negative power.

7. Use the quadratic formula to solve $2x^2 + 3x - 4 = 0$. Give exact answers.
   $a = 2, b = 3, c = -4$. $\Delta = 3^2 - 4(2)(-4) = 9 + 32 = 41$. $x = (-3 \pm \sqrt{41}) \div (2 \times 2) = (-3 \pm \sqrt{41}) \div 4$. So $x = (-3 + \sqrt{41})/4$ or $x = (-3 - \sqrt{41})/4$.
   Achieved: applies the quadratic formula correctly to a non-factorising quadratic.

8. Solve $3^(x+1) = 81$.
   $81 = 3^4$, so $3^(x+1) = 3^4$, giving $x + 1 = 4$, therefore $x = 3$.
   Achieved: solves an exponential equation using equal bases.

9. A rectangular deck at a bach in Raglan has a length 3 m longer than its width. The area of the deck is 40 m². Form an equation and find the dimensions of the deck.
   Let width = x metres, so length = $x + 3$. Area: $x(x + 3) = 40$, which gives $x^2 + 3x - 40 = 0$. Factorise: $(x + 8)(x - 5) = 0$, so $x = -8$ or $x = 5$. A width cannot be negative, so $x = 5$. Width = 5 m, length = 5 + 3 = 8 m. Check: $5 \times 8 = 40$ m². ✓
   Merit: sets up a quadratic model from context and rejects the invalid solution.

10. Write $3/(x - 2) + 2/(x + 1)$ as a single simplified fraction.
    Common denominator is $(x - 2)(x + 1)$. $= [3(x + 1) + 2(x - 2)] \div [(x - 2)(x + 1)] = (3x + 3 + 2x - 4) \div [(x - 2)(x + 1)] = (5x - 1) \div [(x - 2)(x + 1)]$.
    Merit: links equivalent fractions through a common denominator in a multi-step manipulation.

11. Solve $9^x = 27^(x - 2)$.
    Write each side base 3: $9 = 3^2$ and $27 = 3^3$. So $(3^2)^x = (3^3)^(x - 2)$, giving $3^(2x) = 3^(3(x - 2))$. Equate powers: $2x = 3x - 6$, so $6 = x$. Therefore $x = 6$.
    Merit: connects two different bases to a common base to solve.

12. For what values of k does the equation $x^2 + kx + 4 = 0$ have two distinct real roots? Justify your answer.
    Two distinct real roots require the discriminant to be positive: $\Delta = b^2 - 4ac > 0$. Here $a = 1, b = k, c = 4$, so $\Delta = k^2 - 16 > 0$. Then $k^2 > 16$, which means $k < -4$ or $k > 4$. At $k = \pm 4$ there is exactly one repeated root, and for $-4 < k < 4$ there are no real roots. So two distinct real roots occur when $k < -4$ or $k > 4$.
    Excellence: uses the discriminant to reason about and justify a range of parameter values.

13. Solve $\log_2 x + \log_2(x - 2) = 3$, showing why any invalid solution must be rejected.
    Combine the logs: $\log_2[x(x - 2)] = 3$. Rewrite in index form: $x(x - 2) = 2^3 = 8$, so $x^2 - 2x - 8 = 0$. Factorise: $(x - 4)(x + 2) = 0$, giving $x = 4$ or $x = -2$. Check the domain: logs require positive arguments. If $x = -2$, then $\log_2 x = \log_2(-2)$ is

undefined, so $x = -2$ must be rejected. Only $x = 4$ works ($\log_2 4 + \log_2 2 = 2 + 1 = 3$ ✓).

Excellence: solves a log equation and justifies rejecting a solution using domain reasoning.

14. Prove that the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

Let the two consecutive odd numbers be $2n - 1$ and $2n + 1$ (for integer n). Their squares differ by: $(2n + 1)^2 - (2n - 1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$. Since n is an integer, $8n$ is a multiple of 8. Therefore the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

Excellence: uses algebraic generalisation to construct a proof.