NCEA Level 2 · AS 91262: Calculus Methods

Name: _____ Date: _____

This standard is all about differentiation, integration, and using them to solve gradient, tangent, optimisation and motion problems. Work through the sections in order, build fluency first, then push into the Merit and Excellence reasoning.

Revision summary

- Differentiation: if $y = ax^n$ then $dy/dx = anx^{n-1}$. The derivative gives the gradient function.
- Gradient of a curve at $x = a$ is found by substituting a into dy/dx. Tangent line: $y - y_1 = m(x - x_1)$.
- Anti-differentiation (integration): $\int ax^n\,dx = ax^{n+1}/(n+1) + c$ $(n \neq -1)$. Always add $+ c$ for an indefinite integral; find c using a known point.
- Increasing where $dy/dx > 0$, decreasing where $dy/dx < 0$. Turning points where $dy/dx = 0$.
- Second derivative test: $d^2y/dx^2 > 0 \rightarrow$ minimum, $d^2y/dx^2 < 0 \rightarrow$ maximum.
- Kinematics: differentiate displacement $s \rightarrow$ velocity $v \rightarrow$ acceleration a. Integrate $a \rightarrow v \rightarrow s$. At rest means $v = 0$.
- Optimisation: write the quantity as a function of one variable, differentiate, set derivative $= 0$, solve, and check it is a max or min.
- Units: gradient/rate = change in y per unit x; velocity in m s^{-1}, acceleration in m s^{-2}.

Key skills

1. Differentiate $y = 4x^3 - 5x^2 + 7x - 2$.

2. Find $\int (6x^2 + 4x - 3)\,dx$.

3. Find the gradient of the curve $y = x^2 - 6x + 5$ at the point where $x = 4$.

4. A ball's height is $h = 20t - 5t^2$ metres. Find its velocity function.

Achieved

5. Find the equation of the tangent to $y = x^2 - 3x + 1$ at the point where $x = 2$.

6. The gradient function of a curve is $dy/dx = 3x^2 - 4x$. The curve passes through $(1, 2)$. Find y as a function of x.

7. Find the coordinates of the turning point of $y = x^2 - 8x + 10$.

8. A drone rises so its height is $h = t^3 - 6t^2 + 9t$ metres, where t is in seconds. Find its velocity when $t = 1$ s.

Merit

9. A rectangular vegetable garden in Avondale is to be fenced against a straight wall, so only three sides need fencing. There are 60 m of fencing available. Find the dimensions that give the maximum area.

10. A curve has gradient function $dy/dx = 6x - 12$. Find the x-values where the curve is decreasing, and find where the curve has a turning point.

11. A particle moves so its velocity is $v = 3t^2 - 12t + 9$ m s^{-1}. It starts at position $s = 5$ m when $t = 0$. Find its displacement function and the times when it is at rest.

12. The tangent to the curve $y = x^2 + kx - 4$ at the point where $x = 1$ has gradient 5. Find k, then find the equation of that tangent.

Excellence

13. A cylindrical rainwater tank (no lid) is to hold a fixed volume V. Show that the surface area of metal used is least when the height equals the radius.

14. A curve $y = ax^3 + bx^2$ has a turning point at $(2, -4)$. Find a and b, justifying your reasoning.

15. A ball is thrown upward with height $h = ut - 5t^2$ metres (u is the initial speed in m s^{-1}). Prove that the maximum height reached is $u^2/20$ metres, regardless of u.

Answer key (for parents and tutors)

1. Differentiate $y = 4x^3 - 5x^2 + 7x - 2$.
   $dy/dx = 12x^2 - 10x + 7$. (Bring down the power and reduce it by one for each term; the constant -2 differentiates to 0.)
   Achieved: applies the power rule to a polynomial.

2. Find $\int (6x^2 + 4x - 3)\,dx$.
   $\int (6x^2 + 4x - 3)\,dx = 6x^3/3 + 4x^2/2 - 3x + c = 2x^3 + 2x^2 - 3x + c$. Remember the + c.
   Achieved: applies the integration rule and includes the constant.

3. Find the gradient of the curve $y = x^2 - 6x + 5$ at the point where $x = 4$.
   $dy/dx = 2x - 6$. At $x = 4$: gradient $= 2(4) - 6 = 2$.
   Achieved: evaluates the gradient function at a point.

4. A ball's height is $h = 20t - 5t^2$ metres. Find its velocity function.
   $v = dh/dt = 20 - 10t$ (metres per second).
   Achieved: differentiates displacement to obtain velocity.

5. Find the equation of the tangent to $y = x^2 - 3x + 1$ at the point where $x = 2$.
   When $x = 2$: $y = 4 - 6 + 1 = -1$, so the point is $(2, -1)$. $dy/dx = 2x - 3$, so at $x = 2$ the gradient $m = 2(2) - 3 = 1$. Tangent: $y - (-1) = 1(x - 2) \rightarrow y = x - 3$.
   Achieved: finds a point and gradient, then writes the tangent equation.

6. The gradient function of a curve is $dy/dx = 3x^2 - 4x$. The curve passes through $(1, 2)$. Find y as a function of x.
   $y = \int (3x^2 - 4x)\,dx = x^3 - 2x^2 + c$. Substitute $(1, 2)$: $2 = 1 - 2 + c \rightarrow c = 3$. So $y = x^3 - 2x^2 + 3$.
   Achieved: anti-differentiates and uses a point to find c.

7. Find the coordinates of the turning point of $y = x^2 - 8x + 10$.
   $dy/dx = 2x - 8 = 0 \rightarrow x = 4$. Then $y = 16 - 32 + 10 = -6$. Turning point at $(4, -6)$. (It is a minimum since the x^2 coefficient is positive.)
   Achieved: sets the derivative to zero to locate a turning point.

8. A drone rises so its height is $h = t^3 - 6t^2 + 9t$ metres, where t is in seconds. Find its velocity when $t = 1$ s.
   $v = dh/dt = 3t^2 - 12t + 9$. At $t = 1$: $v = 3 - 12 + 9 = 0$ m s^{-1} (the drone is momentarily at rest).
   Achieved: applies differentiation in a kinematics context.

9. A rectangular vegetable garden in Avondale is to be fenced against a straight wall, so only three sides need fencing. There are 60 m of fencing available. Find the dimensions that give the maximum area.
   Let the two sides perpendicular to the wall be x, and the side parallel be y. Fencing: $2x + y = 60$, so $y = 60 - 2x$. Area $A = xy = x(60 - 2x) = 60x - 2x^2$. $dA/dx = 60 - 4x = 0 \rightarrow x = 15$. Then $y = 60 - 30 = 30$. $d^2A/dx^2 = -4 < 0$, so this is a maximum. Dimensions: 15 m by 30 m, giving area = 450 m^2.
   Merit: builds a model, links constraint to area, optimises and verifies it is a maximum.

10. A curve has gradient function $dy/dx = 6x - 12$. Find the x-values where the curve is decreasing, and find where the curve has a turning point.
    The curve decreases where $dy/dx < 0$: $6x - 12 < 0 \rightarrow x < 2$. Turning point where $dy/dx = 0$: $6x - 12 = 0 \rightarrow x = 2$. So the curve decreases for $x < 2$, increases for $x > 2$, with a turning point at $x = 2$ (a minimum, since the gradient changes from negative to positive).
    Merit: links the sign of the derivative to increasing/decreasing behaviour and turning points.

11. A particle moves so its velocity is $v = 3t^2 - 12t + 9$ m s^{-1}. It starts at position $s = 5$ m when $t = 0$. Find its displacement function and the times when it is at rest.
    At rest when $v = 0$: $3t^2 - 12t + 9 = 0 \rightarrow t^2 - 4t + 3 = 0 \rightarrow (t - 1)(t - 3) = 0 \rightarrow t = 1$ s and $t = 3$ s. Displacement: $s = \int v\,dt = t^3 - 6t^2 + 9t + c$. At $t = 0$, $s = 5$, so $c = 5$. Thus $s = t^3 - 6t^2 + 9t + 5$.
    Merit: connects integration with initial conditions and solves for rest points.

12. The tangent to the curve $y = x^2 + kx - 4$ at the point where $x = 1$ has gradient 5. Find k, then find the equation of that tangent.

$dy/dx = 2x + k$. At $x = 1$: $2 + k = 5 \rightarrow k = 3$. So $y = x^2 + 3x - 4$. At $x = 1$: $y = 1 + 3 - 4 = 0$, point $(1, 0)$. Tangent: $y - 0 = 5(x - 1) \rightarrow y = 5x - 5$.

Merit: uses a gradient condition to find an unknown parameter, then constructs the tangent.

13. A cylindrical rainwater tank (no lid) is to hold a fixed volume V. Show that the surface area of metal used is least when the height equals the radius.

    Let radius r, height h. Volume: $V = \pi r^2 h$, so $h = V/(\pi r^2)$. Surface area (base + curved side) $S = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r(V/(\pi r^2)) = \pi r^2 + 2V/r$. $dS/dr = 2\pi r - 2V/r^2$. Set to 0: $2\pi r = 2V/r^2 \rightarrow \pi r^3 = V \rightarrow r^3 = V/\pi$. Since $V = \pi r^2 h$, $r^3 = r^2 h$, so $r = h$. $d^2S/dr^2 = 2\pi + 4V/r^3 > 0$, confirming a minimum. Therefore the surface area is least when $h = r$.

    Excellence: generalises with a symbolic volume and proves the optimal condition.

14. A curve $y = ax^3 + bx^2$ has a turning point at $(2, -4)$. Find a and b, justifying your reasoning.

    $dy/dx = 3ax^2 + 2bx$. At the turning point $x = 2$: $3a(4) + 2b(2) = 0 \rightarrow 12a + 4b = 0 \rightarrow b = -3a$. The point lies on the curve: $-4 = a(8) + b(4) = 8a + 4b$. Substitute $b = -3a$: $-4 = 8a + 4(-3a) = 8a - 12a = -4a \rightarrow a = 1$, so $b = -3$. Curve: $y = x^3 - 3x^2$. Check: $dy/dx = 3x^2 - 6x = 3x(x - 2)$, zero at $x = 2$ ✓, and $y = 8 - 12 = -4$ ✓.

    Excellence: sets up and solves simultaneous conditions from an unfamiliar constraint, with justification.

15. A ball is thrown upward with height $h = ut - 5t^2$ metres (u is the initial speed in m s^{-1}). Prove that the maximum height reached is $u^2/20$ metres, regardless of u.

    Velocity $v = dh/dt = u - 10t$. Maximum height when $v = 0$: $u - 10t = 0 \rightarrow t = u/10$. Substitute into h: $h = u(u/10) - 5(u/10)^2 = u^2/10 - 5(u^2/100) = u^2/10 - u^2/20 = 2u^2/20 - u^2/20 = u^2/20$. Since $d^2h/dt^2 = -10 < 0$, this is a maximum. Therefore the maximum height is $u^2/20$ for any u.

    Excellence: generalises the result algebraically and proves it holds for all values of the parameter.