

NCEA Level 3 · AS 91575: Trig Methods & Periodic Models

Name: _____

Date: _____

Kia ora! This standard is all about solving trig equations (in radians), using compound and double-angle formulae, proving identities, and modelling things that repeat like tides and Ferris wheels. Work top to bottom, nail the Key skills first, then push into the graded questions.

Revision summary

- Radians: $\pi \text{ rad} = 180^\circ$. To convert degrees $\rightarrow$ radians $\times \pi/180$; radians $\rightarrow$ degrees $\times 180/\pi$. Common: $\pi/6 = 30^\circ$, $\pi/4 = 45^\circ$, $\pi/3 = 60^\circ$, $\pi/2 = 90^\circ$.
- Compound angles: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.
- Double angles: $\sin 2\theta = 2\sin\theta \cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
- General solutions: $\sin x = k \rightarrow x = \alpha + 2n\pi$ or $(\pi - \alpha) + 2n\pi$; $\cos x = k \rightarrow x = \pm\alpha + 2n\pi$; $\tan x = k \rightarrow x = \alpha + n\pi$ (n is any integer).
- Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$, so $\tan\theta = \sin\theta/\cos\theta$, $\cot\theta = \cos\theta/\sin\theta$.
- Model form $y = A \sin(B(t - C)) + D$: amplitude $= |A|$, period $= 2\pi/B$, D = vertical shift (midline), C = horizontal shift. Max $= D + A$, min $= D - A$.
- Wave combination: $a \sin x + b \cos x = R \sin(x + \phi)$, where $R = \sqrt{a^2 + b^2}$ and $\tan\phi = b/a$.
- Solving strategy: use identities to get ONE trig ratio, factorise if quadratic-in-trig, solve, then list all solutions in the required interval.

Key skills

1. Convert 135° to radians (exact), and convert $5\pi/6$ radians to degrees.
2. Write down the exact value of $\cos(2\pi/3)$.
3. Use a compound-angle formula to find the exact value of $\sin(75^\circ)$.
4. Simplify $1 - 2\sin^2\theta$ and simplify $2\sin\theta \cos\theta$.

Achieved

5. Solve $2\sin x - 1 = 0$ for $0 \leq x \leq 2\pi$.
6. Write the general solution (in radians) of $\cos\theta = -1/2$.
7. A and B are acute angles with $\sin A = 3/5$ and $\cos B = 5/13$. Find the exact value of $\sin(A + B)$.
8. A Ferris wheel at Ōrākei has radius 25 m and its centre is 30 m above the ground. It completes one revolution every 20 seconds, starting from the lowest point. Its height is $h(t) = 30 - 25\cos(\pi t/10)$ metres. State the amplitude and period, and find the height after 5 seconds.

Merit

9. Solve $2\cos^2x + \cos x - 1 = 0$ for $0 \leq x \leq 2\pi$.
10. Solve $\sin 2x = \cos x$ for $0 \leq x \leq 2\pi$.
11. The depth of water at a Manukau Harbour jetty is modelled by $d = 8 + 3\sin(\pi t/6)$ metres, where t is hours after midnight. A boat needs a depth of at least 9.5 m to enter. During the first 12 hours ($0 \leq t \leq 12$), for what time interval is the depth at least 9.5 m?

Excellence

12. Prove the identity $(1 + \cos 2\theta)/\sin 2\theta = \cot \theta$, and state any values of θ for which it is undefined.
13. Express $3\sin x + 4\cos x$ in the form $R\sin(x + \varphi)$ with $R > 0$ and $0 < \varphi < \pi/2$, then find the general solution of $3\sin x + 4\cos x = 2.5$.
14. Prove that $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$.

Answer key (for parents and tutors)

- Convert 135° to radians (exact), and convert $5\pi/6$ radians to degrees.
 $135^\circ = 135 \times \pi/180 = 3\pi/4$ radians. $5\pi/6 = 5\pi/6 \times 180/\pi = 5 \times 30 = 150^\circ$.
Achieved: fluent conversion between degrees and radians.
- Write down the exact value of $\cos(2\pi/3)$.
 $2\pi/3 = 120^\circ$, which is in the second quadrant. $\cos(120^\circ) = -\cos(60^\circ) = -1/2$.
Achieved: recalls exact values and quadrant sign.
- Use a compound-angle formula to find the exact value of $\sin(75^\circ)$.
 $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2) = (\sqrt{6} + \sqrt{2})/4$.
Achieved: applies the compound-angle expansion.
- Simplify $1 - 2\sin^2\theta$ and simplify $2\sin\theta\cos\theta$.
 $1 - 2\sin^2\theta = \cos 2\theta$ (double-angle form). $2\sin\theta\cos\theta = \sin 2\theta$.
Achieved: recognises double-angle identities.
- Solve $2\sin x - 1 = 0$ for $0 \leq x \leq 2\pi$.
 $2\sin x = 1$, so $\sin x = 1/2$. The reference angle is $\pi/6$. Sine is positive in quadrants 1 and 2, so $x = \pi/6$ and $x = \pi - \pi/6 = 5\pi/6$.
Achieved: selects and applies a method to solve within a stated interval.
- Write the general solution (in radians) of $\cos\theta = -1/2$.
The reference angle is $\pi/3$. Cosine is negative in quadrants 2 and 3, so $\theta = 2\pi/3$ and $\theta = 4\pi/3$ within one revolution. General solution: $\theta = 2\pi/3 + 2n\pi$ or $\theta = 4\pi/3 + 2n\pi$, where n is any integer.
Achieved: produces a correct general solution using period 2π .
- A and B are acute angles with $\sin A = 3/5$ and $\cos B = 5/13$. Find the exact value of $\sin(A + B)$.
Since A is acute, $\cos A = 4/5$. Since B is acute, $\sin B = 12/13$. $\sin(A + B) = \sin A \cos B + \cos A \sin B = (3/5)(5/13) + (4/5)(12/13) = 15/65 + 48/65 = 63/65$.
Achieved: applies compound-angle formula with found ratios.
- A Ferris wheel at Ōrākei has radius 25 m and its centre is 30 m above the ground. It completes one revolution every 20 seconds, starting from the lowest point. Its height is $h(t) = 30 - 25\cos(\pi t/10)$ metres. State the amplitude and period, and find the height after 5 seconds.
Amplitude = 25 m (the coefficient of the cosine). Period = $2\pi \div (\pi/10) = 20$ seconds. At $t = 5$: $h(5) = 30 - 25\cos(\pi \times 5/10) = 30 - 25\cos(\pi/2) = 30 - 25(0) = 30$ m.
Achieved: reads model parameters and evaluates a periodic function.
- Solve $2\cos^2 x + \cos x - 1 = 0$ for $0 \leq x \leq 2\pi$.
Treat as a quadratic in $\cos x$: $(2\cos x - 1)(\cos x + 1) = 0$. So $\cos x = 1/2$ or $\cos x = -1$. $\cos x = 1/2$ gives $x = \pi/3$ and $x = 5\pi/3$. $\cos x = -1$ gives $x = \pi$. Solutions: $x = \pi/3, \pi, 5\pi/3$.
Merit: links factorising with solving a trig equation over an interval.
- Solve $\sin 2x = \cos x$ for $0 \leq x \leq 2\pi$.
Use the double angle: $2\sin x \cos x = \cos x$. Bring to one side: $2\sin x \cos x - \cos x = 0$, so $\cos x(2\sin x - 1) = 0$. Either $\cos x = 0 \rightarrow x = \pi/2, 3\pi/2$; or $\sin x = 1/2 \rightarrow x = \pi/6, 5\pi/6$. Solutions: $x = \pi/6, \pi/2, 5\pi/6, 3\pi/2$. (Do not divide by $\cos x$ or solutions are lost.)
Merit: connects double-angle identity to a multi-step solution and avoids losing roots.
- The depth of water at a Manukau Harbour jetty is modelled by $d = 8 + 3\sin(\pi t/6)$ metres, where t is hours after midnight. A boat needs a depth of at least 9.5 m to enter. During the first 12 hours ($0 \leq t \leq 12$), for what time interval is the depth at least 9.5 m?
Require $8 + 3\sin(\pi t/6) \geq 9.5$, so $3\sin(\pi t/6) \geq 1.5$, giving $\sin(\pi t/6) \geq 0.5$. Let $u = \pi t/6$. $\sin u = 0.5$ at $u = \pi/6$ and $5\pi/6$, and $\sin u \geq 0.5$ between these: $\pi/6 \leq u \leq 5\pi/6$. So $\pi/6 \leq \pi t/6 \leq 5\pi/6$, giving $1 \leq t \leq 5$. The depth is at least 9.5 m from 1:00 am to 5:00 am.
Merit: interprets a model, solves an inequality and translates back to context.

12. Prove the identity $(1 + \cos 2\theta)/\sin 2\theta = \cot \theta$, and state any values of θ for which it is undefined.

Take the left side and use $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$. $\text{LHS} = (1 + 2\cos^2\theta - 1)/(2\sin\theta\cos\theta) = (2\cos^2\theta)/(2\sin\theta\cos\theta) = \cos\theta/\sin\theta = \cot\theta = \text{RHS}$. ■ The identity is undefined where $\sin 2\theta = 0$, i.e. $\theta = n\pi/2$ (any integer n), since that makes the denominator zero (and also where $\sin\theta = 0$ for $\cot\theta$).

Excellence: constructs a logical proof and states domain restrictions.

13. Express $3\sin x + 4\cos x$ in the form $R\sin(x + \varphi)$ with $R > 0$ and $0 < \varphi < \pi/2$, then find the general solution of $3\sin x + 4\cos x = 2.5$.

$R\sin(x + \varphi) = R\cos\varphi \sin x + R\sin\varphi \cos x$. Match: $R\cos\varphi = 3$, $R\sin\varphi = 4$. So $R = \sqrt{3^2 + 4^2} = 5$ and $\tan\varphi = 4/3$, $\varphi = 0.927$ rad. Thus $3\sin x + 4\cos x = 5\sin(x + 0.927)$. Solve $5\sin(x + 0.927) = 2.5 \rightarrow \sin(x + 0.927) = 0.5$. Reference angle $\pi/6 = 0.524$. So $x + 0.927 = 0.524 + 2n\pi$ or $x + 0.927 = (\pi - 0.524) + 2n\pi = 2.618 + 2n\pi$. Hence $x = -0.403 + 2n\pi$ or $x = 1.691 + 2n\pi$, where n is any integer.

Excellence: transforms a wave sum and produces a fully justified general solution.

14. Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Write $3\theta = 2\theta + \theta$ and expand: $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$. Substitute $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$: $= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta = 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$. Now replace $\cos^2\theta = 1 - \sin^2\theta$: $= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta = 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta = 3\sin\theta - 4\sin^3\theta$. ■

Excellence: generalises using compound and double-angle formulae to derive a triple-angle identity.