NCEA Level 3 · AS 91577: Complex Numbers & Their Algebra

Name: _____ Date: _____

Hi! This standard is all about working confidently with complex numbers in both rectangular (a + bi) and polar (r cis θ) form, using De Moivre's theorem, and interpreting loci. Work through the sheet in order, the Key skills warm you up, then push into Merit and Excellence where you'll link ideas and justify your reasoning.

Revision summary

- Rectangular form: $z = a + bi$ where $i^2 = -1$. Conjugate $\bar{z} = a - bi$, and $z \cdot \bar{z} = a^2 + b^2 = |z|^2$.
- Polar form: $z = r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{(a^2 + b^2)}$ and $\theta = \arg(z)$ (choose the quadrant carefully; $-\pi < \theta \leq \pi$).
- Multiply in polar: multiply moduli, add arguments. Divide: divide moduli, subtract arguments. To divide in rectangular form, multiply top and bottom by the conjugate of the denominator.
- De Moivre's theorem: $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ for all rational n.
- nth roots of a complex number: take the nth root of the modulus and use angles $(\theta + 2\pi k)/n$ for $k = 0, 1, \ldots, n-1 \rightarrow n$ roots equally spaced around a circle.
- Loci: $|z - a|$ = distance from the point a; $|z - a| = |z - b|$ is the perpendicular bisector of a and b; $|z - a| = r$ is a circle centre a radius r.
- Surds/indices: $\sqrt{(-n)} = i\sqrt{n}$; rationalise denominators by multiplying by the conjugate; a^(m/n) = ($\sqrt[n]{a}$)^m.

Key skills

1. Simplify $\sqrt{(-48)}$.

2. Write $z = 2 + 2i$ in polar form.

3. Expand and simplify $(3 + 2i)(1 - 4i)$.

4. Use De Moivre's theorem to evaluate $(2\,\text{cis}(\pi/6))^3$, giving your answer in rectangular form.

Achieved

5. Express $(4 + 3i)/(2 - i)$ in the form $a + bi$.

6. Write $z = -\sqrt{3} + i$ in polar form with $-\pi < \theta \leq \pi$.

7. Solve $z^2 = -9$ for z.

8. Rationalise the denominator and simplify: $2/(\sqrt{7} - \sqrt{3})$.

Merit

9. Find all three cube roots of $8i$, giving your answers in rectangular form.

10. Given $z = 1 + i$, use De Moivre's theorem to find z^6 in rectangular form.

11. Describe and identify the locus of points z satisfying $|z - 2| = |z + 2i|$.

Excellence

12. Prove, using mathematical induction, that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

13. Show that the sum of all n distinct nth roots of unity is 0 for $n \geq 2$.

14. A complex number z satisfies $z + 1/z = 2\cos\theta$. Prove that $z^n + 1/z^n = 2\cos n\theta$ for every positive integer n.