NCEA Level 3 · AS 91577: Complex Numbers & Their Algebra

Name: _____ Date: _____

Hi! This standard is all about working confidently with complex numbers in both rectangular (a + bi) and polar (r cis θ) form, using De Moivre's theorem, and interpreting loci. Work through the sheet in order, the Key skills warm you up, then push into Merit and Excellence where you'll link ideas and justify your reasoning.

Revision summary

- Rectangular form: $z = a + bi$ where $i^2 = -1$. Conjugate $\bar{z} = a - bi$, and $z \cdot \bar{z} = a^2 + b^2 = |z|^2$.
- Polar form: $z = r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{(a^2 + b^2)}$ and $\theta = \arg(z)$ (choose the quadrant carefully; $-\pi < \theta \leq \pi$).
- Multiply in polar: multiply moduli, add arguments. Divide: divide moduli, subtract arguments. To divide in rectangular form, multiply top and bottom by the conjugate of the denominator.
- De Moivre's theorem: $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ for all rational n.
- nth roots of a complex number: take the nth root of the modulus and use angles $(\theta + 2\pi k)/n$ for $k = 0, 1, \ldots, n-1 \rightarrow n$ roots equally spaced around a circle.
- Loci: $|z - a|$ = distance from the point a; $|z - a| = |z - b|$ is the perpendicular bisector of a and b; $|z - a| = r$ is a circle centre a radius r.
- Surds/indices: $\sqrt{(-n)} = i\sqrt{n}$; rationalise denominators by multiplying by the conjugate; a^(m/n) = ($\sqrt[n]{a}$)^m.

Key skills

1. Simplify $\sqrt{(-48)}$.

2. Write $z = 2 + 2i$ in polar form.

3. Expand and simplify $(3 + 2i)(1 - 4i)$.

4. Use De Moivre's theorem to evaluate $(2\,\text{cis}(\pi/6))^3$, giving your answer in rectangular form.

Achieved

5. Express $(4 + 3i)/(2 - i)$ in the form $a + bi$.

6. Write $z = -\sqrt{3} + i$ in polar form with $-\pi < \theta \leq \pi$.

7. Solve $z^2 = -9$ for z.

8. Rationalise the denominator and simplify: $2/(\sqrt{7} - \sqrt{3})$.

Merit

9. Find all three cube roots of $8i$, giving your answers in rectangular form.

10. Given $z = 1 + i$, use De Moivre's theorem to find z^6 in rectangular form.

11. Describe and identify the locus of points z satisfying $|z - 2| = |z + 2i|$.

Excellence

12. Prove, using mathematical induction, that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

13. Show that the sum of all n distinct nth roots of unity is 0 for $n \geq 2$.

14. A complex number z satisfies $z + 1/z = 2\cos\theta$. Prove that $z^n + 1/z^n = 2\cos n\theta$ for every positive integer n.

Answer key (for parents and tutors)

1. Simplify $\sqrt{(-48)}$.
   $\sqrt{(-48)} = \sqrt{48} \times \sqrt{(-1)} = \sqrt{(16 \times 3)} \times i = 4\sqrt{3}\,i$.
   Achieved fluency: rewrites the square root of a negative using i.

2. Write $z = 2 + 2i$ in polar form.
   $r = \sqrt{(2^2 + 2^2)} = \sqrt{8} = 2\sqrt{2}$. The point is in the first quadrant, so $\arg = \arctan(2/2) = \pi/4$. Therefore $z = 2\sqrt{2}\,\mathrm{cis}(\pi/4)$.
   Achieved fluency: converts rectangular to polar with correct modulus and argument.

3. Expand and simplify $(3 + 2i)(1 - 4i)$.
   $(3 + 2i)(1 - 4i) = 3 - 12i + 2i - 8i^2 = 3 - 10i + 8 = 11 - 10i$.
   Achieved fluency: multiplies complex numbers using $i^2 = -1$.

4. Use De Moivre's theorem to evaluate $(2\,\mathrm{cis}(\pi/6))^3$, giving your answer in rectangular form.
   $(2\,\mathrm{cis}(\pi/6))^3 = 2^3\,\mathrm{cis}(3 \times \pi/6) = 8\,\mathrm{cis}(\pi/2) = 8(\cos \pi/2 + i \sin \pi/2) = 8(0 + i) = 8i$.
   Achieved fluency: applies De Moivre's theorem to a power.

5. Express $(4 + 3i)/(2 - i)$ in the form $a + bi$.
   Multiply numerator and denominator by the conjugate $(2 + i)$: numerator $= (4 + 3i)(2 + i) = 8 + 4i + 6i + 3i^2 = 8 + 10i - 3 = 5 + 10i$. Denominator $= (2 - i)(2 + i) = 4 + 1 = 5$. So the result $= (5 + 10i)/5 = 1 + 2i$.
   Achieved: selects the conjugate method to divide complex numbers.

6. Write $z = -\sqrt{3} + i$ in polar form with $-\pi < \theta \leq \pi$.
   $r = \sqrt{((-\sqrt{3})^2 + 1^2)} = \sqrt{(3 + 1)} = 2$. The point is in the second quadrant. Reference angle $= \arctan(1/\sqrt{3}) = \pi/6$, so $\arg = \pi - \pi/6 = 5\pi/6$. Therefore $z = 2\,\mathrm{cis}(5\pi/6)$.
   Achieved: converts to polar and places the argument in the correct quadrant.

7. Solve $z^2 = -9$ for z.
   $z^2 = -9$ means $z = \pm\sqrt{(-9)} = \pm\sqrt{9} \times i = \pm 3i$. So $z = 3i$ or $z = -3i$.
   Achieved: solves a simple quadratic giving complex roots.

8. Rationalise the denominator and simplify: $2/(\sqrt{7} - \sqrt{3})$.
   Multiply top and bottom by the conjugate $(\sqrt{7} + \sqrt{3})$: $2(\sqrt{7} + \sqrt{3}) / ((\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})) = 2(\sqrt{7} + \sqrt{3})/(7 - 3) = 2(\sqrt{7} + \sqrt{3})/4 = (\sqrt{7} + \sqrt{3})/2$.
   Achieved: applies surd manipulation by rationalising a denominator.

9. Find all three cube roots of $8i$, giving your answers in rectangular form.
   Write $8i$ in polar form: $8i = 8\,\mathrm{cis}(\pi/2)$. Cube roots have modulus $\sqrt[3]{8} = 2$ and arguments $(\pi/2 + 2\pi k)/3$ for $k = 0, 1, 2$. $k = 0$: $\arg = \pi/6 \to 2\,\mathrm{cis}(\pi/6) = 2(\sqrt{3}/2 + i/2) = \sqrt{3} + i$. $k = 1$: $\arg = \pi/6 + 2\pi/3 = 5\pi/6 \to 2\,\mathrm{cis}(5\pi/6) = 2(-\sqrt{3}/2 + i/2) = -\sqrt{3} + i$. $k = 2$: $\arg = \pi/6 + 4\pi/3 = 3\pi/2 \to 2\,\mathrm{cis}(3\pi/2) = 2(0 - i) = -2i$. Roots: $\sqrt{3} + i$, $-\sqrt{3} + i$, $-2i$.
   Merit: links polar form, De Moivre and root spacing across multiple steps.

10. Given $z = 1 + i$, use De Moivre's theorem to find z^6 in rectangular form.
    Convert: $r = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $\arg = \pi/4$, so $z = \sqrt{2}\,\mathrm{cis}(\pi/4)$. Then $z^6 = (\sqrt{2})^6\,\mathrm{cis}(6 \times \pi/4) = 8\,\mathrm{cis}(3\pi/2) = 8(\cos 3\pi/2 + i \sin 3\pi/2) = 8(0 - i) = -8i$.
    Merit: connects conversion to polar with De Moivre to evaluate a high power efficiently.

11. Describe and identify the locus of points z satisfying $|z - 2| = |z + 2i|$.
    Let $z = x + iy$. Then $|z - 2|^2 = (x - 2)^2 + y^2$ and $|z + 2i|^2 = x^2 + (y + 2)^2$. Setting them equal: $(x - 2)^2 + y^2 = x^2 + (y + 2)^2$. Expand: $x^2 - 4x + 4 + y^2 = x^2 + y^2 + 4y + 4$. Simplify: $-4x = 4y$, so $y = -x$. The locus is the straight line $y = -x$, which is the perpendicular bisector of the segment joining the points 2 (i.e. $2 + 0i$) and $-2i$ (i.e. $0 - 2i$).
    Merit: connects the algebraic condition to a geometric locus and identifies it.

12. Prove, using mathematical induction, that $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$ for all positive integers n.
    Base case $n = 1$: $(\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta = \cos(1 \cdot \theta) + i \sin(1 \cdot \theta)$. True. Inductive step: assume it holds for $n = k$, i.e. $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$. Then $(\cos \theta + i \sin \theta)^{(k+1)} = (\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)$. Expanding: $= \cos k\theta \cos \theta + i \cos k\theta \sin \theta + i \sin k\theta \cos \theta + i^2 \sin k\theta \sin \theta = (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$. Using the compound-angle formulae: $= \cos(k\theta + \theta) + i \sin(k\theta + \theta) = \cos((k+1)\theta)$

+ i sin((k+1)θ). So it holds for n = k + 1. By induction it holds for all positive integers n.
Excellence: constructs a rigorous inductive proof using compound-angle identities.

13. Show that the sum of all n distinct nth roots of unity is 0 for n ≥ 2.
The nth roots of unity are ω^k for k = 0, 1, …, n−1, where ω = cis(2π/n). These are the roots of $z^n - 1 = 0$. Their sum is a geometric series: S = 1 + ω + ω^2 + … + ω^(n−1) = (ω^n − 1)/(ω − 1). Since ω = cis(2π/n), ω^n = cis(2π) = 1, so the numerator is 0. Because n ≥ 2, ω ≠ 1, so the denominator is non-zero. Therefore S = 0/(ω − 1) = 0. (Equivalently, by Vieta's formulas the sum of roots of $z^n - 1$ equals the negative of the coefficient of z^(n−1), which is 0.)
Excellence: generalises and justifies a result about roots of unity using summation or Vieta's.

14. A complex number z satisfies z + 1/z = 2cos θ. Prove that $z^n + 1/z^n$ = 2cos nθ for every positive integer n.
From z + 1/z = 2cos θ, multiply by z: z^2 − 2cos θ · z + 1 = 0. Solving: z = (2cos θ ± √(4$\cos^2\theta$ − 4))/2 = cos θ ± √($\cos^2\theta$ − 1) = cos θ ± √(−$\sin^2\theta$) = cos θ ± i sin θ. Take z = cos θ + i sin θ = cis θ (the other root gives the same result by symmetry). By De Moivre's theorem, z^n = cis(nθ) = cos nθ + i sin nθ. Also $1/z^n$ = z^(−n) = cis(−nθ) = cos nθ − i sin nθ. Adding: $z^n + 1/z^n$ = (cos nθ + i sin nθ) + (cos nθ − i sin nθ) = 2cos nθ. Hence proven for all positive integers n.
Excellence: solves an unfamiliar problem by deducing polar form and generalising with De Moivre.