NCEA Level 3 · AS 91578: Differentiation Toolkit

Name: _____ Date: _____

Hi! This standard is all about differentiating trickier functions (using product, quotient and chain rules, plus trig, e^x and ln) and then using those derivatives to solve real problems, rates, optimisation, tangents, curve shapes and motion. Work through the sheet top to bottom: warm up on the key skills, then build toward the Merit and Excellence reasoning.

Revision summary

- Rules: product $(uv)' = u'v + uv'$; quotient $(u/v)' = (u'v - uv')/v^2$; chain $dy/dx = dy/du \cdot du/dx$.
- Standard derivatives: $d/dx(\sin x)=\cos x$, $d/dx(\cos x)=-\sin x$, $d/dx(\tan x)=\sec^2 x$, $d/dx(e^x)=e^x$, $d/dx(\ln x)=1/x$.
- Chain versions: $d/dx(e^{(f)})=f' \cdot e^{(f)}$; $d/dx(\ln f)=f'/f$; $d/dx(\sin f)=f' \cdot \cos f$.
- Stationary points: solve $f'(x)=0$. Classify with sign of $f'(x)$ or with $f''(x)$: $f''>0$ minimum, $f''<0$ maximum, test if $f''=0$.
- Optimisation: write the quantity in one variable using a constraint, differentiate, set derivative = 0, then justify it is a max/min.
- Related rates: differentiate with respect to t; chain the rates together, e.g. $dV/dt = dV/dr \cdot dr/dt$.
- Kinematics: velocity $v = ds/dt$, acceleration $a = dv/dt = d^2s/dt^2$. Object at rest when $v = 0$.
- Tangent at $x = a$: gradient $m = f'(a)$, line $y - f(a) = m(x - a)$.

Key skills

1. Differentiate $y = (3x^2 + 1)^5$.

2. Differentiate $y = x^2 e^x$.

3. Differentiate $y = (2x + 1)/(x - 3)$.

4. Differentiate $y = \ln(\sin x)$.

Achieved

5. Find the gradient of the curve $y = 3\sin(2x)$ at $x = \pi/6$.

6. Find the coordinates of the stationary points of $y = x^3 - 6x^2 + 9x + 2$.

7. A trolley moves so its position is $s = t^3 - 4t^2 + 2t$ metres after t seconds. Find its velocity at $t = 3$ s.

8. Find the equation of the tangent to $y = e^{(2x)}$ at the point where $x = 0$.

Merit

9. A farmer near Pukekohe uses 120 m of fencing to make a rectangular paddock against a straight river. No fence is needed along the river. Find the dimensions that give the maximum area, and state that maximum area.

10. A closed cylindrical can must hold 500 cm³. Find the radius that minimises the surface area (give it to 2 d.p.).

11. A spherical weather balloon is inflated so its radius increases at 2 cm/s. Find the rate at which its volume is increasing when the radius is 5 cm.

Excellence

12. For the curve $y = x^3 - 3kx + 2$, determine all values of the constant k for which the curve has two distinct stationary points. Justify your answer.

13. Using calculus, prove that among all rectangles with a fixed perimeter P, the square gives the greatest area.

14. A particle moves along a line with displacement $x(t) = e^{(-t)} \sin t$ metres for $t \geq 0$. Show that the particle is instantaneously at rest whenever $\tan t = 1$, and explain why the exponential factor never causes it to stop.

Answer key (for parents and tutors)

1. Differentiate $y = (3x^2 + 1)^5$.
   Use the chain rule with $u = 3x^2 + 1$, so $y = u^5$. $dy/du = 5u^4$ and $du/dx = 6x$. $dy/dx = 5(3x^2 + 1)^4 \times 6x = 30x(3x^2 + 1)^4$.
   Key skill: applies the chain rule to a power of a polynomial.

2. Differentiate $y = x^2e^x$.
   Product rule with $u = x^2$, $v = e^x$. $u' = 2x$, $v' = e^x$. $dy/dx = u'v + uv' = 2x \cdot e^x + x^2 \cdot e^x = e^x(x^2 + 2x) = x\,e^x(x + 2)$.
   Key skill: applies the product rule combining a polynomial and e^x.

3. Differentiate $y = (2x + 1)/(x - 3)$.
   Quotient rule with $u = 2x + 1$, $v = x - 3$. $u' = 2$, $v' = 1$. $dy/dx = (u'v - uv')/v^2 = [2(x - 3) - (2x + 1)(1)]/(x - 3)^2 = (2x - 6 - 2x - 1)/(x - 3)^2 = -7/(x - 3)^2$.
   Key skill: applies the quotient rule and simplifies.

4. Differentiate $y = \ln(\sin x)$.
   Chain rule for a log: $d/dx(\ln f) = f'/f$ with $f = \sin x$, $f' = \cos x$. $dy/dx = \cos x / \sin x = \cot x$.
   Key skill: differentiates a logarithm of a trig function using the chain rule.

5. Find the gradient of the curve $y = 3\sin(2x)$ at $x = \pi/6$.
   $dy/dx = 3 \times \cos(2x) \times 2 = 6\cos(2x)$. At $x = \pi/6$: $6\cos(2 \times \pi/6) = 6\cos(\pi/3) = 6 \times 0.5 = 3$. The gradient is 3.
   Achieved: selects the chain rule for a trig function and evaluates at a point.

6. Find the coordinates of the stationary points of $y = x^3 - 6x^2 + 9x + 2$.
   $dy/dx = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. Set = 0: $x = 1$ or $x = 3$. $y(1) = 1 - 6 + 9 + 2 = 6$; $y(3) = 27 - 54 + 27 + 2 = 2$. Stationary points are (1, 6) and (3, 2).
   Achieved: finds and evaluates stationary points from $f'(x) = 0$.

7. A trolley moves so its position is $s = t^3 - 4t^2 + 2t$ metres after t seconds. Find its velocity at $t = 3$ s.
   Velocity $v = ds/dt = 3t^2 - 8t + 2$. At $t = 3$: $v = 3(9) - 8(3) + 2 = 27 - 24 + 2 = 5$. The velocity is 5 m/s.
   Achieved: differentiates a position function to obtain velocity at a time.

8. Find the equation of the tangent to $y = e^{(2x)}$ at the point where $x = 0$.
   $dy/dx = 2e^{(2x)}$. At $x = 0$ the gradient is $2e^0 = 2$. The point is $(0, e^0) = (0, 1)$. Tangent: $y - 1 = 2(x - 0)$, so $y = 2x + 1$.
   Achieved: finds a tangent line using the derivative of an exponential.

9. A farmer near Pukekohe uses 120 m of fencing to make a rectangular paddock against a straight river. No fence is needed along the river. Find the dimensions that give the maximum area, and state that maximum area.
   Let the two sides perpendicular to the river be x and the side parallel be y. Fencing used: $2x + y = 120$, so $y = 120 - 2x$. Area $A = xy = x(120 - 2x) = 120x - 2x^2$. $dA/dx = 120 - 4x = 0 \rightarrow x = 30$, so $y = 120 - 60 = 60$. $A'' = -4 < 0$, confirming a maximum. Dimensions 30 m × 60 m; maximum area = $30 \times 60 = 1800$ m².
   Merit: links a constraint to the area function and justifies the maximum.

10. A closed cylindrical can must hold 500 cm³. Find the radius that minimises the surface area (give it to 2 d.p.).
    Volume: $\pi r^2 h = 500$, so $h = 500/(\pi r^2)$. Surface area $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r(500/(\pi r^2)) = 2\pi r^2 + 1000/r$. $dS/dr = 4\pi r - 1000/r^2$. Set = 0: $4\pi r^3 = 1000 \rightarrow r^3 = 250/\pi \approx 79.577 \rightarrow r \approx 4.30$ cm. Since dS/dr changes from negative to positive (or $S'' = 4\pi + 2000/r^3 > 0$), this is a minimum. Minimising radius ≈ 4.30 cm.
    Merit: connects volume constraint to surface area and solves the optimisation.

11. A spherical weather balloon is inflated so its radius increases at 2 cm/s. Find the rate at which its volume is increasing when the radius is 5 cm.
    $V = (4/3)\pi r^3$, so $dV/dr = 4\pi r^2$. By the chain rule $dV/dt = dV/dr \times dr/dt = 4\pi r^2 \times 2$. At $r = 5$: $dV/dt = 4\pi(25)(2) = 200\pi \approx 628.3$ cm³/s.
    Merit: chains related rates together using the chain rule.

12. For the curve $y = x^3 - 3kx + 2$, determine all values of the constant k for which the curve has two distinct stationary points. Justify your answer.
    $dy/dx = 3x^2 - 3k$. Stationary points occur where $3x^2 - 3k = 0$, i.e. $x^2 = k$. This equation has two distinct real solutions $x = \pm\sqrt{k}$ only when $k > 0$ (if $k = 0$ there is a single repeated solution $x = 0$, a stationary point of inflection; if

$k < 0$ there are no real solutions). Therefore the curve has two distinct stationary points exactly when $k > 0$.
Excellence: generalises a condition on a parameter and justifies each case.

13. Using calculus, prove that among all rectangles with a fixed perimeter P, the square gives the greatest area.
Let one side be x, so the adjacent side is $P/2 - x$ (from $2x + 2(\text{other}) = P$). Area $A(x) = x(P/2 - x) = (P/2)x - x^2$, valid for $0 < x < P/2$. $A'(x) = P/2 - 2x$. Setting $A'(x) = 0$ gives $x = P/4$, and then the other side $= P/2 - P/4 = P/4$, so the rectangle is a square. $A''(x) = -2 < 0$, so this stationary point is a maximum. Hence the square maximises the area for a given perimeter.
Excellence: proves a general optimisation result with full justification.

14. A particle moves along a line with displacement $x(t) = e^{(-t)} \sin t$ metres for $t \geq 0$. Show that the particle is instantaneously at rest whenever $\tan t = 1$, and explain why the exponential factor never causes it to stop.
Velocity: $x'(t) = d/dt[e^{(-t)} \sin t]$. Product rule: $x'(t) = (-e^{(-t)})\sin t + e^{(-t)}\cos t = e^{(-t)}(\cos t - \sin t)$. The particle is at rest when $x'(t) = 0$. Since $e^{(-t)} > 0$ for all real t, it is never zero, so it can never make the velocity zero on its own. Thus $x'(t) = 0$ requires $\cos t - \sin t = 0$, i.e. $\cos t = \sin t$, i.e. $\tan t = 1$. This occurs at $t = \pi/4, 5\pi/4, 9\pi/4, \ldots$ ($t = \pi/4 + n\pi$). Because $e^{(-t)}$ stays strictly positive, the only way the velocity vanishes is through the trig factor, confirming the condition $\tan t = 1$.
Excellence: differentiates an unfamiliar product and reasons about why a factor cannot be zero.