NCEA Level 3 · AS 91579: Integration Methods

Name: _____ Date: _____

Hi! This standard is all about anti-derivatives, finding areas, using substitution, and solving differential equations. Work through each section in order, the questions get harder as you go, so use the early ones to warm up your fluency before tackling the Merit and Excellence problems.

Revision summary

- Power rule: $\int x^n\,dx = x^{n+1}/(n+1) + c$, for $n \neq -1$. Always add + c for indefinite integrals.
- Standard forms: $\int e^{kx}\,dx = (1/k)e^{kx} + c$; $\int (1/x)\,dx = \ln|x| + c$; $\int \sin(kx)\,dx = -(1/k)\cos(kx) + c$; $\int \cos(kx)\,dx = (1/k)\sin(kx) + c$.
- Definite integral (area): $\int$ from a to b of f(x) dx = F(b) − F(a). Area below the x-axis counts as negative, so split the integral at the roots for total area.
- Substitution: let u = inner function, then du = (du/dx) dx. Replace everything in x with u (including limits for definite integrals) before integrating.
- Differential equations: dy/dx = f(x) is solved by integrating, y = ∫f(x) dx + c. Use a given point (initial condition) to find c.
- Rate problems: if dV/dt is given, then V = ∫(dV/dt) dt; the constant represents the starting amount.
- Chain-rule check: to integrate f(g(x))·g'(x), the derivative of the inner function must appear (up to a constant) as a factor.
- Area between curves: ∫ from a to b of (top curve − bottom curve) dx, where a and b are the intersection points.

Key skills

1. Find $\int (6x^2 - 4x + 5)\,dx$.

2. Find $\int (e^{3x} + 2\cos x)\,dx$.

3. Evaluate $\int$ from 1 to 3 of $(2x)\,dx$.

4. Find $\int (4/x)\,dx$ for $x > 0$.

Achieved

5. The gradient function of a curve is $dy/dx = 3x^2 - 8x$. The curve passes through the point (2, 1). Find the equation of the curve.

6. Find the area between the curve $y = x^2 + 1$ and the x-axis from $x = 0$ to $x = 2$.

7. Use the substitution $u = x^2 + 3$ to find $\int 2x(x^2 + 3)^4\,dx$.

8. Water flows into a tank at a rate of $dV/dt = 6t + 2$ litres per minute, where t is in minutes. If the tank starts empty, find the volume of water after 5 minutes.

Merit

9. Find the area enclosed between the curves $y = x^2$ and $y = 6x - x^2$.

10. Find $\int x\sqrt{(2x^2 + 1)}\,dx$ using a suitable substitution, showing your choice.

11. The acceleration of a drone rising vertically is $a = 12 - 6t$ m s^{-2} at time t seconds. At $t = 0$ its velocity is 5 m s^{-1} and its height is 0 m. Find the height when the drone's velocity first returns to 5 m s^{-1}.

Excellence

12. A curve has gradient function $dy/dx = kx + 1$, where k is a constant. The curve passes through (0, 3) and (2, 9). Find the value of k and the equation of the curve, justifying your working.

13. Show that $\int$ from 0 to a of $(2x)\,dx = \int$ from 0 to a of $(3x^2)\,dx$ has exactly one positive solution for a, and find it.

14. A population grows so that $dP/dt = 0.4P$, where P is the number of insects and t is in days. Given that $P = 200$ when $t = 0$, show that $P = 200e^{(0.4t)}$, and find how long it takes the population to reach 1000. (Hint: rewrite as $(1/P)\,dP = 0.4\,dt$ and integrate both sides.)

Answer key (for parents and tutors)

1. Find $\int(6x^2 - 4x + 5)\,dx$.
   Integrate term by term: $6x^2 \to 6x^3/3 = 2x^3$; $-4x \to -4x^2/2 = -2x^2$; $5 \to 5x$. So $\int(6x^2 - 4x + 5)\,dx = 2x^3 - 2x^2 + 5x + c$.
   Achieved: applies the power rule to a polynomial.

2. Find $\int(e^{3x} + 2\cos x)\,dx$.
   $\int e^{3x}\,dx = (1/3)e^{3x}$; $\int 2\cos x\,dx = 2\sin x$. So the answer is $(1/3)e^{3x} + 2\sin x + c$.
   Achieved: recalls exponential and trig anti-derivatives.

3. Evaluate $\int$ from 1 to 3 of $(2x)\,dx$.
   Anti-derivative of $2x$ is x^2. Evaluate: $(3^2) - (1^2) = 9 - 1 = 8$.
   Achieved: applies the fundamental theorem to a definite integral.

4. Find $\int(4/x)\,dx$ for $x > 0$.
   $\int(4/x)\,dx = 4\int(1/x)\,dx = 4\ln|x| + c = 4\ln x + c$ (since $x > 0$).
   Achieved: recognises the 1/x standard form giving a logarithm.

5. The gradient function of a curve is $dy/dx = 3x^2 - 8x$. The curve passes through the point $(2, 1)$. Find the equation of the curve.
   Integrate: $y = \int(3x^2 - 8x)\,dx = x^3 - 4x^2 + c$. Substitute $(2, 1)$: $1 = 2^3 - 4(2^2) + c = 8 - 16 + c = -8 + c$, so $c = 9$. The curve is $y = x^3 - 4x^2 + 9$.
   Achieved: integrates a gradient function and uses a point to find c.

6. Find the area between the curve $y = x^2 + 1$ and the x-axis from $x = 0$ to $x = 2$.
   Area = $\int$ from 0 to 2 of $(x^2 + 1)\,dx = [x^3/3 + x]$ from 0 to 2 $= (8/3 + 2) - (0) = 8/3 + 6/3 = 14/3 \approx 4.67$ square units.
   Achieved: sets up and evaluates a definite integral for area.

7. Use the substitution $u = x^2 + 3$ to find $\int 2x(x^2 + 3)^4\,dx$.
   Let $u = x^2 + 3$, so $du = 2x\,dx$. The integral becomes $\int u^4\,du = u^5/5 + c$. Substitute back: $(x^2 + 3)^5/5 + c$.
   Achieved: applies a given substitution correctly.

8. Water flows into a tank at a rate of $dV/dt = 6t + 2$ litres per minute, where t is in minutes. If the tank starts empty, find the volume of water after 5 minutes.
   $V = \int(6t + 2)\,dt = 3t^2 + 2t + c$. Tank starts empty: at $t = 0$, $V = 0$, so $c = 0$. Thus $V = 3t^2 + 2t$. At $t = 5$: $V = 3(25) + 2(5) = 75 + 10 = 85$ litres.
   Achieved: integrates a rate and applies an initial condition.

9. Find the area enclosed between the curves $y = x^2$ and $y = 6x - x^2$.
   Find intersections: $x^2 = 6x - x^2 \to 2x^2 - 6x = 0 \to 2x(x - 3) = 0$, so $x = 0$ and $x = 3$. Between these, $6x - x^2$ is the top curve. Area = $\int$ from 0 to 3 of $((6x - x^2) - x^2)\,dx = \int$ from 0 to 3 of $(6x - 2x^2)\,dx = [3x^2 - 2x^3/3]$ from 0 to 3 $= (3(9) - 2(27)/3) - 0 = (27 - 18) = 9$ square units.
   Merit: links intersection points and the difference of functions to find enclosed area.

10. Find $\int x\sqrt{(2x^2 + 1)}\,dx$ using a suitable substitution, showing your choice.
    Let $u = 2x^2 + 1$, so $du = 4x\,dx$, meaning $x\,dx = du/4$. Then $\int x\sqrt{(2x^2 + 1)}\,dx = \int\sqrt{u} \cdot (1/4)\,du = (1/4)\int u\text{^}(1/2)\,du = (1/4)\cdot(u\text{^}(3/2)/(3/2)) + c = (1/4)\cdot(2/3)u\text{^}(3/2) + c = (1/6)(2x^2 + 1)\text{^}(3/2) + c$.
    Merit: chooses an appropriate substitution and works through multiple steps.

11. The acceleration of a drone rising vertically is $a = 12 - 6t$ m s^{-2} at time t seconds. At $t = 0$ its velocity is 5 m s^{-1} and its height is 0 m. Find the height when the drone's velocity first returns to 5 m s^{-1}.
    Velocity: $v = \int(12 - 6t)\,dt = 12t - 3t^2 + c_1$. At $t = 0$, $v = 5$, so $c_1 = 5$, giving $v = 12t - 3t^2 + 5$. Height: $h = \int v\,dt = 6t^2 - t^3 + 5t + c_2$. At $t = 0$, $h = 0$, so $c_2 = 0$, giving $h = 6t^2 - t^3 + 5t$. Velocity returns to 5 when $12t - 3t^2 + 5 = 5 \to 12t - 3t^2 = 0 \to 3t(4 - t) = 0$, so $t = 4$ (excluding $t = 0$). Height at $t = 4$: $h = 6(16) - 64 + 20 = 96 - 64 + 20 = 52$ m.
    Merit: integrates twice, links acceleration→velocity→displacement with conditions.

12. A curve has gradient function $dy/dx = kx + 1$, where k is a constant. The curve passes through $(0, 3)$ and $(2, 9)$. Find the value of k and the equation of the curve, justifying your working.
    Integrate: $y = \int(kx + 1)\,dx = kx^2/2 + x + c$. Through $(0, 3)$: $3 = 0 + 0 + c$, so $c = 3$. Through $(2, 9)$: $9 = k(4)/2 + 2 + 3 = 2k$

+ 5, so 2k = 4, k = 2. The equation is $y = x^2 + x + 3$. Check: $dy/dx = 2x + 1 = kx + 1$ with k = 2. ✓ Both points satisfy: (0,3) and (2, 4+2+3=9). ✓

Excellence: solves for an unknown constant within an integrated model and justifies with both conditions.

13. Show that ∫ from 0 to a of (2x) dx = ∫ from 0 to a of ($3x^2$) dx has exactly one positive solution for a, and find it.
Left side: $[x^2]$ from 0 to $a = a^2$. Right side: $[x^3]$ from 0 to $a = a^3$. Set equal: $a^2 = a^3 \rightarrow a^3 - a^2 = 0 \rightarrow a^2(a - 1) = 0$, so a = 0 or a = 1. The only positive solution is a = 1. Justification: for $a > 0$, $a^2(a - 1) = 0$ requires a = 1, since $a^2 > 0$; hence exactly one positive value. Both integrals then equal 1.

Excellence: forms and justifies an equation from two definite integrals and argues uniqueness.

14. A population grows so that dP/dt = 0.4P, where P is the number of insects and t is in days. Given that P = 200 when t = 0, show that P = 200e^(0.4t), and find how long it takes the population to reach 1000. (Hint: rewrite as (1/P) dP = 0.4 dt and integrate both sides.)
Separate: (1/P) dP = 0.4 dt. Integrate both sides: ∫(1/P) dP = ∫0.4 dt → ln P = 0.4t + c (P > 0). So P = e^(0.4t + c) = Ae^(0.4t) where A = e^c. At t = 0, P = 200: $200 = A \cdot e^0 = A$, so A = 200. Therefore P = 200e^(0.4t). To reach 1000: 1000 = 200e^(0.4t) → 5 = e^(0.4t) → ln 5 = 0.4t → t = ln 5 / 0.4 ≈ 1.609/0.4 ≈ 4.02 days.

Excellence: solves a separable differential equation from first principles and interprets the result.