NCEA Level 3 · AS 91586: Working with Probability Distributions

Name: Date:

Hi! This standard is all about choosing the right probability model (normal, binomial, Poisson, uniform or triangular) and using it to answer real questions. Work through each section in order, check your working against the answers, and pay attention to when you have to justify your choice of model, that's where the top grades live.

Revision summary

- Normal: standardise with $z = (x - \mu)/\sigma$, then use tables/calculator for $P(Z < z)$. Inverse: given a probability, find z first, then $x = \mu + z\sigma$.
- Binomial $X \sim B(n, p)$: fixed n independent trials, constant p. $P(X = k) = C(n,k)\, p^k (1-p)^{(n-k)}$; mean = np, variance = np(1−p). Use when counting successes in a set number of trials.
- Poisson $X \sim Po(\lambda)$: events at a constant average rate over a fixed interval. $P(X = k) = e^{(-\lambda)} \lambda^k / k!$; mean = variance = λ. Scale λ in proportion to the interval length.
- Uniform (continuous on [a,b]): constant density $1/(b-a)$; $P(c < X < d) = (d-c)/(b-a)$; mean = (a+b)/2.
- Triangular(a, b, c) with min a, max b, mode c: peak height $2/(b-a)$. $P(X \le c) = (c-a)/(b-a)$. Areas are triangles, so use ½ × base × height.
- Model choice: binomial = fixed trials with success/failure; Poisson = counts over time/space with a rate (use as approx to binomial when n large, p small, λ = np); normal = continuous, symmetric, bell-shaped data.
- Useful z-values: $P(Z<z)=0.90 \rightarrow z=1.2816$; $0.95 \rightarrow 1.6449$; $0.975 \rightarrow 1.96$; 0.02 lower tail → z=−2.054.
- Normal approximation to binomial ($np \ge 5$ and $n(1-p) \ge 5$): $\mu = np$, $\sigma = \sqrt{(np(1-p))}$, and apply a continuity correction of ±0.5.

Key skills

1. X is normally distributed with mean 50 and standard deviation 8. Find $P(X < 60)$.

2. $X \sim$ Binomial($n = 10$, $p = 0.3$). Find $P(X = 4)$.

3. The number of texts Aroha gets per hour follows a Poisson distribution with mean 3. Find $P(X = 2)$.

4. A bus arrival delay is uniformly distributed on the interval [0, 20] minutes. Find $P(5 < X < 12)$.

Achieved

5. The weight of a kererū is normally distributed with mean 650 g and standard deviation 40 g. Find the probability a randomly chosen kererū weighs more than 700 g.

6. A 12-question multiple-choice quiz has four options per question. Manu guesses every answer, so the chance each is correct is 0.25. Find the probability he gets exactly 3 correct.

7. A café receives online orders at an average rate of 5 per hour, following a Poisson distribution. Find the probability that fewer than 3 orders arrive in a given hour.

8. Delivery times for a courier follow a triangular distribution with minimum 20 min, mode 30 min and maximum 50 min. Find the probability a delivery takes less than 30 minutes.

Merit

9. A machine fills juice bottles so the volume is normally distributed with standard deviation 5 mL. The manager wants only 2% of bottles to fall below the labelled 500 mL. What mean fill volume should be set?

10. A factory line produces components that are defective 1.5% of the time, independently. A batch of 200 is checked. Explain why a Poisson model is suitable and find the probability that at most 2 components in the batch are defective.

11. Bags of chips have weights that are normally distributed with mean 150 g and standard deviation 4 g. In a shipment of 500 bags, how many would you expect to weigh between 145 g and 155 g?

Excellence

12. A test score X is normally distributed. It is known that 10% of students score below 40 and 5% score above 75. Find the mean and standard deviation of the distribution.

13. Minor earthquakes felt in Auckland occur at an average rate of 0.8 per year, modelled as Poisson. Find the probability that at least 5 are felt over a 5–year period, and state one assumption the model relies on.

14. A fair coin is tossed 400 times. Using an appropriate distribution, estimate the probability of getting at least 210 heads, and justify why your chosen method is valid.

Answer key (for parents and tutors)

1. X is normally distributed with mean 50 and standard deviation 8. Find P(X < 60).
   Standardise: $z = (60 - 50)/8 = 1.25$. $P(Z < 1.25) = 0.8944$. So $P(X < 60) \approx 0.894$.
   Achieved: standardises and reads a normal probability.

2. X ~ Binomial(n = 10, p = 0.3). Find P(X = 4).
   $P(X = 4) = C(10,4) \times 0.3^4 \times 0.7^6 = 210 \times 0.0081 \times 0.117649 = 0.2001$ (4 dp).
   Achieved: applies the binomial probability formula.

3. The number of texts Aroha gets per hour follows a Poisson distribution with mean 3. Find P(X = 2).
   $P(X = 2) = e^{-3} \times 3^2 / 2! = 0.049787 \times 9 / 2 = 0.2240$.
   Achieved: applies the Poisson probability formula.

4. A bus arrival delay is uniformly distributed on the interval [0, 20] minutes. Find P(5 < X < 12).
   For a continuous uniform distribution, $P(5 < X < 12) = (12 - 5)/(20 - 0) = 7/20 = 0.35$.
   Achieved: uses the uniform distribution to find a probability.

5. The weight of a kererū is normally distributed with mean 650 g and standard deviation 40 g. Find the probability a randomly chosen kererū weighs more than 700 g.
   $z = (700 - 650)/40 = 1.25$. $P(Z > 1.25) = 1 - 0.8944 = 0.1056$. So $P(X > 700) \approx 0.106$.
   Achieved: selects and applies the normal model to an upper tail.

6. A 12-question multiple-choice quiz has four options per question. Manu guesses every answer, so the chance each is correct is 0.25. Find the probability he gets exactly 3 correct.
   $X \sim B(12, 0.25)$. $P(X = 3) = C(12,3) \times 0.25^3 \times 0.75^9 = 220 \times 0.015625 \times 0.075085 = 0.258$.
   Achieved: identifies binomial conditions and computes a probability.

7. A café receives online orders at an average rate of 5 per hour, following a Poisson distribution. Find the probability that fewer than 3 orders arrive in a given hour.
   $\lambda = 5$. $P(X < 3) = P(0) + P(1) + P(2) = e^{-5}(1 + 5 + 12.5) = 0.0067379 \times 18.5 = 0.1247$.
   Achieved: applies the Poisson model to a cumulative 'fewer than' event.

8. Delivery times for a courier follow a triangular distribution with minimum 20 min, mode 30 min and maximum 50 min. Find the probability a delivery takes less than 30 minutes.
   For Triangular(a=20, c=30, b=50), $P(X \le \text{mode}) = (c - a)/(b - a) = (30 - 20)/(50 - 20) = 10/30 = 0.333$.
   Achieved: applies the triangular distribution to find an area up to the mode.

9. A machine fills juice bottles so the volume is normally distributed with standard deviation 5 mL. The manager wants only 2% of bottles to fall below the labelled 500 mL. What mean fill volume should be set?
   We need $P(X < 500) = 0.02$, so the z-value for the 0.02 lower tail is $z = -2.054$. Then $500 = \mu + (-2.054)(5)$, so $\mu = 500 + 10.27 = 510.3$ mL. The machine should be set to about 510.3 mL.
   Merit: works backwards from a tail probability using inverse normal to find a parameter.

10. A factory line produces components that are defective 1.5% of the time, independently. A batch of 200 is checked. Explain why a Poisson model is suitable and find the probability that at most 2 components in the batch are defective.
    With $n = 200$ large and $p = 0.015$ small, the binomial can be approximated by a Poisson with $\lambda = np = 200 \times 0.015 = 3$. $P(X \le 2) = e^{-3}(1 + 3 + 3^2/2) = e^{-3}(1 + 3 + 4.5) = 0.049787 \times 8.5 = 0.423$.
    Merit: links binomial conditions to a Poisson approximation and applies it.

11. Bags of chips have weights that are normally distributed with mean 150 g and standard deviation 4 g. In a shipment of 500 bags, how many would you expect to weigh between 145 g and 155 g?
    z-values: $(145 - 150)/4 = -1.25$ and $(155 - 150)/4 = 1.25$. $P(-1.25 < Z < 1.25) = 0.8944 - 0.1056 = 0.7888$. Expected number = $0.7888 \times 500 = 394.4$, so about 394 bags.
    Merit: links a normal probability to an expected frequency over many trials.

12. A test score X is normally distributed. It is known that 10% of students score below 40 and 5% score above 75. Find the mean and standard deviation of the distribution.
    For the lower 10%, $z = -1.2816$, giving $40 = \mu - 1.2816\sigma$. For the upper 5%, $z = 1.6449$, giving $75 = \mu + 1.6449\sigma$.

Subtracting the equations: 75 − 40 = (1.6449 + 1.2816)σ, so 35 = 2.9265σ, giving σ = 11.96. Then μ = 40 + 1.2816 × 11.96 = 55.3. So μ ≈ 55.3 and σ ≈ 12.0.

Excellence: sets up and solves simultaneous equations from two inverse-normal conditions.

13. Minor earthquakes felt in Auckland occur at an average rate of 0.8 per year, modelled as Poisson. Find the probability that at least 5 are felt over a 5-year period, and state one assumption the model relies on.
Over 5 years, λ = 0.8 × 5 = 4. P(X ≥ 5) = 1 − P(X ≤ 4). P(0)=e^(−4)=0.018316, P(1)=0.073263, P(2)=0.146525, P(3)=0.195367, P(4)=0.195367; sum = 0.628838. So P(X ≥ 5) = 1 − 0.6288 = 0.371. Assumption: earthquakes occur independently at a constant average rate, questionable, since aftershocks cluster, which could make the Poisson model less valid.

Excellence: scales the rate, evaluates a cumulative tail, and critically evaluates a model assumption.

14. A fair coin is tossed 400 times. Using an appropriate distribution, estimate the probability of getting at least 210 heads, and justify why your chosen method is valid.
Exactly, X ~ B(400, 0.5). Since np = 200 and n(1−p) = 200 are both well above 5, a normal approximation is valid: μ = np = 200, σ = √(np(1−p)) = √100 = 10. Apply a continuity correction: P(X ≥ 210) ≈ P(Y ≥ 209.5) where Y ~ N(200, 10^2). z = (209.5 − 200)/10 = 0.95. P(Z > 0.95) = 1 − 0.8289 = 0.1711. So P ≈ 0.171. The approximation is justified because n is large and p is not extreme, so the binomial is close to symmetric and bell-shaped.

Excellence: justifies a normal approximation to the binomial and applies a continuity correction.