NCEA Level 3 · AS 91587: Systems of Simultaneous Equations

Name: _____ Date: _____

This standard is all about solving and modelling problems with three unknowns using Gaussian elimination, and knowing when a system has one solution, infinitely many, or none. Work through the sections in order, the Key skills warm up your method, then push into the Merit and Excellence reasoning that earns the top grades.

Revision summary

- A system of 3 equations in 3 unknowns (x, y, z) usually represents three planes; the solution is where they all meet.
- Gaussian elimination: use row operations (swap rows, multiply a row by a non-zero number, add a multiple of one row to another) to reach upper-triangular form, then back-substitute.
- Unique solution: elimination gives a leading variable in every row (three planes meet at one point).
- Infinite solutions: a row becomes 0 = 0 (a redundant equation); write the answer parametrically, e.g. let z = t and express x and y in terms of t.
- No solution: a row becomes 0 = k where k ≠ 0 (a contradiction, the planes have no common point).
- To model: define each unknown clearly, turn each condition into one linear equation, then solve and interpret the answer in context.
- For a system with a parameter, isolate the parameter in the final row: (parameter)·variable = value decides unique / infinite / none.
- Always check a solution by substituting back into every original equation.

Key skills

1. Solve this triangular system by back-substitution: $x + 2y - z = 3$; $y + 2z = 5$; $z = 2$.

2. Use elimination to solve: $2x + y - z = 3$; $x - y + 2z = 1$; $x + y + z = 4$.

3. A system reduces so that its final row becomes: (a) $0 = 0$, (b) $0 = 7$, (c) a leading variable in each of the three rows. State the number of solutions in each case.

4. Write the solution set in parametric form: $x + y + z = 5$; $x + 2y + 3z = 8$; $2x + 3y + 4z = 13$.

Achieved

5. Solve the system: $x + y + z = 100$; $2x + y + 3z = 180$; $x + 2y + z = 150$.

6. At a Avondale café, three orders are placed. Order 1: 2 pies, 3 coffees and 1 muffin cost $26. Order 2: 1 pie, 2 coffees and 2 muffins cost $20. Order 3: 3 pies, 1 coffee and 1 muffin cost $20. Find the price of each item.

7. Determine whether the system has a unique solution, infinitely many, or none: $x + y + z = 4$; $2x + 2y + 2z = 8$; $x - y + z = 2$. If infinite, give the solution parametrically.

8. $10 000 is split between three accounts earning 3%, 4% and 5% per year. The total interest for the year is $370, and the amount in the first account equals the total of the other two. How much is in each account?

Merit

9. A concert sells adult, student and child tickets at $25, $15 and $10. A total of 500 tickets are sold for total revenue of $9000, and the number of students is twice the number of children. Find how many of each ticket were sold.

10. A parabola $y = ax^2 + bx + c$ passes through the points (1, 6), (2, 11) and (3, 18). Find its equation, connecting the points to a system of equations.

11. Reduce and interpret: $x + 2y - z = 3$; $2x - y + 3z = 4$; $x - 3y + 4z = 1$. State the solution type and give the general solution.

Excellence

12. For the system $x + y + z = 6$; $x + 2y + 3z = 10$; $x + 2y + kz = m$, find the value(s) of k and m for which the system has (i) a unique solution, (ii) infinitely many solutions, (iii) no solution. Justify your answer.

13. Investigate the system $ax + y + z = 1$; $x + ay + z = 1$; $x + y + az = 1$. Determine all values of a giving a unique solution, infinitely many, or none, and justify.

14. Balancing the combustion reaction $a\cdot C_3H_8 + b\cdot O_2 \rightarrow c\cdot CO_2 + d\cdot H_2O$ leads to a system for a, b, c, d. Set up the equations, show it has infinitely many solutions, and explain how a valid balanced equation is chosen.

Answer key (for parents and tutors)

1. Solve this triangular system by back-substitution: $x + 2y - z = 3$; $y + 2z = 5$; $z = 2$.
   $z = 2$. Substitute into the second equation: $y + 2(2) = 5$, so $y = 1$. Substitute into the first: $x + 2(1) - 2 = 3$, so $x = 3$. Solution: $(x, y, z) = (3, 1, 2)$.
   Achieved: applies back-substitution to a triangular system.

2. Use elimination to solve: $2x + y - z = 3$; $x - y + 2z = 1$; $x + y + z = 4$.
   From equation 3, $x + y + z = 4$. Add equation 2 and equation 3: $(x + x) + (-y + y) + (2z + z) = 1 + 4 \rightarrow 2x + 3z = 5$... instead eliminate y. Equation 1 + equation 2: $3x + z = 4$. Equation 3 + equation 2: $2x + 3z = 5$. From $3x + z = 4$, $z = 4 - 3x$. Substitute: $2x + 3(4 - 3x) = 5 \rightarrow 2x + 12 - 9x = 5 \rightarrow -7x = -7 \rightarrow x = 1$. Then $z = 4 - 3 = 1$, and from equation 3, $y = 4 - 1 - 1 = 2$. Solution: $(1, 2, 1)$.
   Achieved: carries out Gaussian elimination on a 3×3 system.

3. A system reduces so that its final row becomes: (a) $0 = 0$, (b) $0 = 7$, (c) a leading variable in each of the three rows. State the number of solutions in each case.
   (a) $0 = 0$ is a redundant equation, so there are infinitely many solutions. (b) $0 = 7$ is a contradiction, so there is no solution. (c) A leading variable in every row means each unknown is determined, so there is a unique solution.
   Achieved: classifies solution type from row-echelon form.

4. Write the solution set in parametric form: $x + y + z = 5$; $x + 2y + 3z = 8$; $2x + 3y + 4z = 13$.
   Notice equation 3 = equation 1 + equation 2 ($2x + 3y + 4z = 13$), so it is redundant → infinitely many solutions. Equation 2 − equation 1: $y + 2z = 3$, so $y = 3 - 2z$. From equation 1: $x = 5 - y - z = 5 - (3 - 2z) - z = 2 + z$. Let $z = t$. Solution: $(x, y, z) = (2 + t, 3 - 2t, t)$, for any real t.
   Achieved: expresses an infinite solution set with a parameter.

5. Solve the system: $x + y + z = 100$; $2x + y + 3z = 180$; $x + 2y + z = 150$.
   Equation 3 − equation 1: $y = 50$. Equation 2 − 2×equation 1: $(2x - 2x) + (y - 2y) + (3z - 2z) = 180 - 200 \rightarrow -y + z = -20$, so $z = y - 20 = 30$. From equation 1: $x = 100 - 50 - 30 = 20$. Solution: $(x, y, z) = (20, 50, 30)$. Check in equation 2: $40 + 50 + 90 = 180$ ✓.
   Achieved: selects and applies elimination to reach a unique solution.

6. At a Avondale café, three orders are placed. Order 1: 2 pies, 3 coffees and 1 muffin cost $26. Order 2: 1 pie, 2 coffees and 2 muffins cost $20. Order 3: 3 pies, 1 coffee and 1 muffin cost $20. Find the price of each item.
   Let p, c, m be the prices. $2p + 3c + m = 26$; $p + 2c + 2m = 20$; $3p + c + m = 20$. Equation 1 − equation 3: $-p + 2c = 6$. Equation 1 − 2×(equation 2)? Instead: multiply equation 2 by 2: $2p + 4c + 4m = 40$, subtract equation 1: $c + 3m = 14$. Multiply equation 3 by 2: $6p + 2c + 2m = 40$, subtract 3×equation 2 ($3p + 6c + 6m = 60$): $3p - 4c - 4m = -20$. Simpler: from $-p + 2c = 6$, $p = 2c - 6$. Substitute into equation 3: $3(2c - 6) + c + m = 20 \rightarrow 7c + m = 38$. Substitute p into equation 2: $(2c - 6) + 2c + 2m = 20 \rightarrow 4c + 2m = 26 \rightarrow 2c + m = 13$. Subtract: $(7c + m) - (2c + m) = 38 - 13 \rightarrow 5c = 25 \rightarrow c = 5$. Then $m = 13 - 10 = 3$, and $p = 2(5) - 6 = 4$. Pies $4, coffees $5, muffins $3.
   Achieved: models a straightforward context and solves the system.

7. Determine whether the system has a unique solution, infinitely many, or none: $x + y + z = 4$; $2x + 2y + 2z = 8$; $x - y + z = 2$. If infinite, give the solution parametrically.
   Equation 2 is exactly 2×equation 1, so it is redundant. Two independent equations remain in three unknowns → infinitely many solutions. Equation 1 − equation 3: $2y = 2$, so $y = 1$. Then $x + z = 3$. Let $z = t$: $x = 3 - t$. Solution: $(3 - t, 1, t)$ for any real t.
   Achieved: identifies a dependent system and parametrises it.

8. $10 000 is split between three accounts earning 3%, 4% and 5% per year. The total interest for the year is $370, and the amount in the first account equals the total of the other two. How much is in each account?
   Let a, b, c be the amounts. $a + b + c = 10\,000$; $0.03a + 0.04b + 0.05c = 370$ (×100: $3a + 4b + 5c = 37\,000$); $a = b + c \rightarrow a - b - c = 0$. Add equation 1 and the third: $2a = 10\,000$, $a = 5000$, so $b + c = 5000$. Substitute into $3a + 4b + 5c = 37\,000$: $15\,000 + 4b + 5c = 37\,000 \rightarrow 4b + 5c = 22\,000$. With $b = 5000 - c$: $4(5000 - c) + 5c = 22\,000 \rightarrow 20\,000 + c = 22\,000 \rightarrow c = 2000$, $b = 3000$. Accounts: $5000, $3000, $2000.
   Achieved: translates a financial context into equations and solves.

9. A concert sells adult, student and child tickets at $25, $15 and $10. A total of 500 tickets are sold for total revenue of $9000, and the number of students is twice the number of children. Find how many of each ticket were sold.
Let a, s, c be the numbers. a + s + c = 500; 25a + 15s + 10c = 9000; s = 2c. Substitute s = 2c into the first: a + 3c = 500, so a = 500 − 3c. Substitute into the revenue equation: 25(500 − 3c) + 15(2c) + 10c = 9000 → 12 500 − 75c + 30c + 10c = 9000 → 12 500 − 35c = 9000 → 35c = 3500 → c = 100. Then s = 200 and a = 500 − 300 = 200. Adults 200, students 200, children 100. Check revenue: 5000 + 3000 + 1000 = 9000 ✓.
Merit: links a stated relationship into the system and solves a multi-step problem.

10. A parabola $y = ax^2 + bx + c$ passes through the points (1, 6), (2, 11) and (3, 18). Find its equation, connecting the points to a system of equations.
Each point gives one equation: a + b + c = 6; 4a + 2b + c = 11; 9a + 3b + c = 18. Equation 2 − equation 1: 3a + b = 5. Equation 3 − equation 2: 5a + b = 7. Subtract: 2a = 2, so a = 1. Then b = 5 − 3 = 2, and c = 6 − 1 − 2 = 3. The parabola is $y = x^2 + 2x + 3$. Check (3, 18): 9 + 6 + 3 = 18 ✓.
Merit: connects a geometric representation (points on a curve) to a solved system.

11. Reduce and interpret: x + 2y − z = 3; 2x − y + 3z = 4; x − 3y + 4z = 1. State the solution type and give the general solution.
Equation 2 − 2×equation 1: (2x − 2x) + (−y − 4y) + (3z + 2z) = 4 − 6 → −5y + 5z = −2. Equation 3 − equation 1: (x − x) + (−3y − 2y) + (4z + z) = 1 − 3 → −5y + 5z = −2. The two reduced equations are identical, so one equation is redundant → infinitely many solutions (the planes meet in a line). Let z = t: from −5y + 5z = −2, y = t + 2/5. From equation 1: x = 3 − 2y + z = 3 − 2(t + 2/5) + t = 11/5 − t. General solution: (11/5 − t, 2/5 + t, t).
Merit: reduces the system and interprets the redundancy geometrically as a line.

12. For the system x + y + z = 6; x + 2y + 3z = 10; x + 2y + kz = m, find the value(s) of k and m for which the system has (i) a unique solution, (ii) infinitely many solutions, (iii) no solution. Justify your answer.
Equation 2 − equation 1: y + 2z = 4. Equation 3 − equation 1: y + (k − 1)z = m − 6. Subtract these two: (k − 1 − 2)z = (m − 6) − 4 → (k − 3)z = m − 10. (i) If k ≠ 3, we can divide by (k − 3) to find z, then y, then x, a unique solution for any m. (ii) If k = 3, the last row becomes 0 = m − 10. This is consistent only when m = 10, giving 0 = 0, a redundant row → infinitely many solutions. (iii) If k = 3 and m ≠ 10, the last row is 0 = (non-zero), a contradiction → no solution.
Excellence: generalises solution behaviour in terms of two parameters with justification.

13. Investigate the system ax + y + z = 1; x + ay + z = 1; x + y + az = 1. Determine all values of a giving a unique solution, infinitely many, or none, and justify.
Equation 1 − equation 2: (a − 1)x − (a − 1)y = 0 → (a − 1)(x − y) = 0. Equation 2 − equation 3: (a − 1)(y − z) = 0. Case a ≠ 1: then x = y = z. Substitute into any equation: (a + 2)x = 1. If a ≠ −2, x = 1/(a + 2) → unique solution x = y = z = 1/(a + 2). If a = −2, this becomes 0 = 1, a contradiction → no solution. Case a = 1: all three equations become x + y + z = 1 (identical), so two are redundant → infinitely many solutions (a whole plane). Summary: unique for a ≠ 1 and a ≠ −2; no solution for a = −2; infinitely many for a = 1.
Excellence: analyses a symmetric parametric system, proving each solution case.

14. Balancing the combustion reaction $a\cdot C_3H_8 + b\cdot O_2 \rightarrow c\cdot CO_2 + d\cdot H_2O$ leads to a system for a, b, c, d. Set up the equations, show it has infinitely many solutions, and explain how a valid balanced equation is chosen.
Match atoms on each side. Carbon: 3a = c. Hydrogen: 8a = 2d → d = 4a. Oxygen: 2b = 2c + d = 2(3a) + 4a = 10a → b = 5a. So c, d, b are all fixed multiples of a, but a itself is free, this is a homogeneous-type system with one free parameter, hence infinitely many solutions (any scaling works). A valid chemical equation needs the smallest positive whole numbers, so choose the smallest a making all values integers: a = 1 gives b = 5, c = 3, d = 4. The balanced equation is $C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$. The infinite solutions correspond to scaling the whole equation (e.g. a = 2 doubles everything).
Excellence: models an unfamiliar context, recognises the infinite-solution structure and justifies the chosen solution.